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On the Integrals of the Squares of Ellipsoidal Surface Harmonic Functions

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V. *On the Integrals of the Squares of Ellipsoidal Surface Harmonic Functions.*

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This paper forms a sequel to three others published in Series A of the ‘Philosophical Transactions,’ namely, “On Ellipsoidal Harmonic Analysis,” vol. 197, pp. 461–557, “On the Pear-shaped Figure of Equilibrium of a Rotating Mass of Liquid,” vol. 198, pp. 301–331, and “On the Stability of the Pear-shaped Figure of Equilibrium, &c.,” vol. 200, pp. 251–314. I shall refer to these three papers as “Harmonics,” “The Pear-shaped Figure,” and “Stability.”

In “Harmonics,” the functions being expressed approximately, approximate formulæ are found for the integrals over the surface of the ellipsoid of the squares of all the surface harmonics. These integrals are of course required whenever it is proposed to make practical use of this method of analysis, and the evaluation of them is therefore an absolutely essential step towards any applications.

The analysis used in the determination of some of these integrals was very complicated, and is probably susceptible of improvement. Such improvement might perhaps be obtained by the methods of the present paper, but I do not care to spend a great deal of time on an attempt merely to improve the analysis.

In “Harmonics” the symmetry which really subsists between the three factors of the solid harmonic functions was sacrificed with the object of obtaining convenient approximate forms, and I do not think it would have been possible to obtain such satisfactory results without this sacrifice.* But this course had the disadvantage of rendering it difficult to evaluate the integrals of the squares of the surface harmonics.

All the harmonic functions up to the third order inclusive are susceptible of rigorous algebraic expression; and indeed the same is true of some but not of all the functions of the fourth order. Accordingly in these cases rigorous expressions for the integrals should also be obtainable, and the object of the present paper is to complete the preceding investigation in this respect.

It will be well to begin by a restatement of the notation. That used in “Harmonics” was convenient for the approximate and asymmetrical expressions

* See Appendix, below.

involved, but the notation used in the two later papers seems preferable where the formulæ are rigorous and symmetrical.

In "Harmonics" the squares of the semi-axes of the ellipsoid were

$$a^2 = k^2\left(\nu^2 - \frac{1+\beta}{1-\beta}\right),\ b^2 = k^2(\nu^2 - 1),\ c^2 = k^2\nu^2.$$

The rectangular coordinates were connected with ellipsoidal coordinates ν, μ, ϕ by

$$\frac{x^2}{k^2} = -\frac{1-\beta}{1+\beta}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)\left(\mu^2 - \frac{1+\beta}{1-\beta}\right)\cos^2\phi,$$

$$\frac{y^2}{k^2} = -(\nu^2 - 1)(\mu^2 - 1)\sin^2\phi,$$

$$\frac{z^2}{k^2} = \nu^2\mu^2\frac{1-\beta\cos 2\phi}{1+\beta}$$

The three roots of the cubic

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1,$$

were

$$u_1 = k^2\nu^2,\ u_2 = k^2\mu^2,\ u_3 = k^2\frac{1-\beta\cos 2\phi}{1-\beta}.$$

Lastly ν ranges from ∞ to 0, μ between ± 1, ϕ from 0 to 2π.

In the two later papers I put

$$\kappa^2 = \frac{1-\beta}{1+\beta},\ \kappa'^2 = 1 - \kappa^2,\ \nu = \frac{1}{\kappa\sin\gamma},\ \mu = \sin\theta;$$

and for convenience I introduced an auxiliary constant β (easily distinguishable from the β of the previous notation) defined by $\sin\beta = \kappa\sin\gamma$.

The squares of the semi-axes of the ellipsoid were then

$$a^2 = \frac{k^2\cos^2\gamma}{\sin^2\beta},\ b^2 = \frac{k^2\cos^2\beta}{\sin^2\beta},\ c^2 = \frac{k^2}{\sin^2\beta}.$$

The rectangular coordinates became

$$\frac{x^2}{k^2} = \frac{\cos^2\gamma}{\sin^2\beta}(1 - \kappa^2\sin^2\theta)\cos^2\phi,\ \frac{y^2}{k^2} = \frac{\cos^2\beta}{\sin^2\beta}\cos^2\theta\sin^2\phi,\ \frac{z^2}{k^2} = \frac{1}{\sin^2\beta}\sin^2\theta(1 - \kappa'^2\cos^2\phi).$$

The roots of the cubic were

$$u_1 = \frac{k^2}{\sin^2\beta},\ u_2 = k^2\sin^2\theta,\ u_3 = \frac{k^2}{\kappa^2}(1 - \kappa'^2\cos^2\phi).$$

This is the notation which will be used in the present paper.

If $d\sigma$ be an element of surface of the ellipsoid, and p the central perpendicular on to the tangent plane, it appears from the formula at the foot of p. 257 of "Stability" that

$$\frac{p\,d\sigma}{d\theta\,d\phi} = \frac{k^3\cos\beta\cos\gamma}{\sin^3\beta}\cdot\frac{\kappa^2\cos^2\theta + \kappa'^2\sin^2\phi}{\Delta\Gamma},$$

where $\Delta^2 = 1 - \kappa^2\sin^2\theta$, $\Gamma^2 = 1 - \kappa'^2\cos^2\phi$.

In the previous papers I have expressed the two factors of which a surface harmonic consists by $\mathfrak{P}_i^s(\mu)$ or $\mathsf{P}_i^s(\mu)$, and $\mathfrak{C}_i^s(\phi)$, $\mathsf{C}_i^s(\phi)$, $\mathfrak{S}_i^s(\phi)$ or $\mathsf{S}_i^s(\phi)$, one of the two P-functions being multiplied by one of the four cosine or sine functions.

Taking a pair of typical cases, the integrals to be evaluated are

$$\int(\mathfrak{P}_i^s\,\mathfrak{C}_i^s)^2 p\,d\sigma \qquad\text{and}\qquad \int(\mathfrak{P}_i^s\,\mathfrak{S}_i^s)^2 p\,d\sigma.$$

As it will be convenient to use an abridged notation, I will write these integrals $I_i^s(\cos)$ and $I_i^s(\sin)$, according to an easily intelligible notation.

These functions involve integrals of even functions, and therefore we may integrate through one octant of space, the limits of θ and ϕ being $\frac{1}{2}\pi$ to 0, and multiply the result by 8.

It is clear then that

$$I_i^s(\cos) = \frac{8k^3\cos\beta\cos\gamma}{\sin^3\beta}\left[\int_0^{\frac{1}{2}\pi}\frac{\kappa^2\cos^2\theta\,(\mathfrak{P}_i^s)^2}{\Delta}\,d\theta\int_0^{\frac{1}{2}\pi}\frac{(\mathfrak{C}_i^s)^2}{\Gamma}\,d\phi + \int_0^{\frac{1}{2}\pi}\frac{(\mathfrak{P}_i^s)^2}{\Delta}\,d\theta\int_0^{\frac{1}{2}\pi}\frac{\kappa'^2\sin^2\phi\,(\mathfrak{C}_i^s)^2}{\Gamma}\,d\phi\right].$$

Similar expressions are applicable to all the other forms of function, but we may proceed with this form as a type of all the others.

This formula shows that the variables are separable, and since we might substitute $\frac{1}{2}\pi - \psi$ for ϕ without changing the result, the ϕ integrals are of the same type as the θ integrals.

It has been stated above that two of the roots of the cubic equation are proportional to $\kappa^2\sin^2\theta$ and $(1 - \kappa'^2\cos^2\phi)$. By the nature of the harmonic functions it follows that if $[\mathfrak{P}_i^s(\mu)]^2$ is proportional to a certain function of $\kappa^2\sin^2\theta$, $[\mathfrak{C}_i^s(\phi)]^2$ is proportional to the same function of $(1 - \kappa'^2\cos^2\phi)$.

It follows that if $(\mathfrak{P}_i^s)^2 = F(\kappa^2 - \kappa^2\sin^2\theta) = F(\kappa^2\cos^2\theta)$,

$$(\mathfrak{C}_i^s)^2 = \alpha F(\kappa^2 - 1 + \kappa'^2\cos^2\phi) = \alpha F(-\kappa'^2\sin^2\phi),$$

where α is a constant, which for the present we may regard as being unity. If then

$$[\mathfrak{P}_i^s(\mu)]^2 = A_0 + A_1\kappa^2\cos^2\theta + A_2\kappa^4\cos^4\theta + A_3\kappa^6\cos^6\theta + \dots,$$

we must have

$$[\mathfrak{C}_i^s(\phi)]^2 = A_0 - A_1\kappa'^2\sin^2\phi + A_2\kappa'^4\sin^4\phi - A_3\kappa'^6\sin^6\phi + \dots$$

Accordingly if there is a term $A_n\int\kappa^{2n+2}\cos^{2n+2}\theta\frac{d\theta}{\Delta}$ in $\int\kappa^2\cos^2\theta\,(\mathfrak{P}_i^s)^2\frac{d\theta}{\Delta}$ and a term $(-)^m A_m\int\kappa'^{2m}\sin^{2m}\phi\frac{d\phi}{\Gamma}$ in $\int(\mathfrak{C}_i^s)^2\frac{d\phi}{\Gamma}$, then there must be a term $A_m\int\kappa^{2m}\cos^{2m}\theta\frac{d\theta}{\Delta}$ in $\int(\mathfrak{P}_i^s)^2\frac{d\theta}{\Delta}$, and a term $(-)^n A_n\int\kappa'^{2n+2}\sin^{2n+2}\phi\frac{d\phi}{\Gamma}$. It follows that the coefficient of $(-)^m A_n A_m$ in $I_i^s(\cos)\div\frac{8k^3\cos\beta\cos\gamma}{\sin^3\beta}$ is

$$\int\kappa^{2n+2}\cos^{2n+2}\theta\frac{d\theta}{\Delta}\int\kappa'^{2m}\sin^{2m}\phi\frac{d\phi}{\Gamma}-(-)^{n-m+1}\int\kappa^{2m}\cos^{2m}\theta\frac{d\theta}{\Delta}\int\kappa'^{2n+2}\sin^{2n+2}\phi\frac{d\phi}{\Gamma}.$$

For the sake of brevity I call this function $[2n+2,\ 2m]$, and we may state that one term in the required expression is $(-)^m A_n A_m[2n+2,\ 2m]$, where [ ] indicates the above function of the four integrals. It follows that $I_i^s(\cos)\div\frac{8k^3\cos\beta\cos\gamma}{\sin^3\beta}=$

$$\left.\begin{array}{l} A_0^2[2,0]-A_0A_1[2,2]+A_0A_2[2,4]-\ldots \\ +A_1A_0[4,0]-A_1^2[4,2]+A_1A_2[4,4]-\ldots \\ +A_2A_0[6,0]-A_2A_1[6,2]+A_2^2[6,4]-\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}\quad\ldots\ldots\quad(1).$$

Since

$$[2n,\ 2m]=\int\kappa^{2n}\cos^{2n}\theta\frac{d\theta}{\Delta}\int\kappa'^{2m}\sin^{2m}\phi\frac{d\phi}{\Gamma}-(-)^{n-m}\int\kappa^{2m}\sin^{2m}\theta\frac{d\theta}{\Delta}\int\kappa'^{2n}\sin^{2n}\phi\frac{d\phi}{\Gamma},$$

it is clear that

$$[2n,\ 2m]=-(-)^{n-m}[2m,\ 2n].$$

Hence if n and m differ by an odd number $[2n,\ 2m]=[2m,\ 2n]$, and if they differ by an even number $[2n,\ 2m]=-[2m,\ 2n]$. Also $[2n,\ 2n]=0$.

Let us write $\{2n\}=\int_0^{\frac{1}{2}\pi}\kappa^{2n}\cos^{2n}\theta\frac{d\theta}{\Delta}$, $\{2n\}'=\int_0^{\frac{1}{2}\pi}\kappa'^{2n}\sin^{2n}\phi\frac{d\phi}{\Gamma}$, so that

$$[2n,\ 2m]=\{2n\}\{2m\}'-(-)^{n-m}\{2n\}'\{2m\}.$$

We must now evaluate these functions.

Since $\Delta^2=1-\kappa^2\sin^2\theta$, we have by differentiation

$$\frac{d}{d\theta}[\Delta\sin\theta\cos^{2n-3}\theta]=$$
$$\frac{1}{\Delta}\{(2n-1)\kappa^2\cos^{2n}\theta-(2n-2)(\kappa^2-\kappa'^2)\cos^{2n-2}\theta-(2n-3)\kappa'^2\cos^{2n-4}\theta\}.$$

Integrating between $\frac{1}{2}\pi$ and 0 and multiplying by κ^{2n-2} we have

$$\{2n\}=\frac{2n-2}{2n-1}(\kappa^2-\kappa'^2)\{2n-2\}+\frac{2n-3}{2n-1}\kappa^2\kappa'^2\{2n-4\},$$

and by symmetry

$$\{2n\}' = -\frac{2n-2}{2n-1}(\kappa^2 - \kappa'^2)\{2n-2\}' + \frac{2n-3}{2n-1}\kappa^2\kappa'^2\{2n-4\}'.$$

Multiplying the first of these by $\{2m\}'$ and the second by $-(-)^{n-m}\{2m\}$ and adding together we have

$$[2n, 2m] = \frac{2n-2}{2n-1}(\kappa^2 - \kappa'^2)[2n-2, 2m] + \frac{2n-3}{2n-1}\kappa^2\kappa'^2[2n-4, 2m].$$

By successive applications of this formula we may reduce any function $[2n, 2m]$ until it depends on $[2, 0]$, but the result becomes very complicated after a few successive reductions.

Now $\{0\} = \int_0^{\frac{1}{2}\pi}\frac{d\theta}{\Delta} = \text{F}, \quad \{0\}' = \int_0^{\frac{1}{2}\pi}\frac{d\theta}{\Gamma} = \text{F}'$

$$\{2\} = \int_0^{\frac{1}{2}\pi}\frac{\kappa^2\cos^2\theta}{\Delta}d\theta = \text{E} - \text{F}\kappa'^2, \quad \{2\}' = \int_0^{\frac{1}{2}\pi}\frac{\kappa'^2\sin^2\phi}{\Gamma}d\phi = \text{E}' - \text{F}'\kappa^2.$$

Then
$$[2, 0] = \{2\}\{0\}' - (-)^1\{2\}'\{0\}$$
$$= \text{EF}' + \text{F}'\text{F} - \text{FF}'.$$

But it is well known that this combination of the complete elliptic integrals with moduli κ and κ' is $\frac{1}{2}\pi$.*

Hence $[2, 0] = \frac{1}{2}\pi$.

It seems unnecessary to reproduce the simple algebra involved in the successive reductions, and I therefore merely give the results, as follows:—

$[0, 2] = [2, 0] = \frac{1}{2}\pi$

$-[0, 4] = [4, 0] = \frac{2}{3}(\kappa^2 - \kappa'^2).\frac{1}{2}\pi$

$[0, 6] = [6, 0] = \frac{1}{3.5}(8 - 23\kappa^2\kappa'^2).\frac{1}{2}\pi$

$-[0, 8] = [8, 0] = \frac{8}{3.5.7}(\kappa^2 - \kappa'^2)(6 - 11\kappa^2\kappa'^2).\frac{1}{2}\pi$

$[2, 4] = [4, 2] = \frac{1}{3}\kappa^2\kappa'^2.\frac{1}{2}\pi$

$-[2, 6] = [6, 2] = \frac{4}{3.5}(\kappa^2 - \kappa'^2)\kappa^2\kappa'^2.\frac{1}{2}\pi$

$[2, 8] = [8, 2] = \frac{1}{3.5.7}(24 - 71\kappa^2\kappa'^2)\kappa^2\kappa'^2.\frac{1}{2}\pi$

$[4, 6] = [6, 4] = \frac{1}{5}\kappa^4\kappa'^4.\frac{1}{2}\pi$

$-[4, 8] = [8, 4] = \frac{6}{5.7}(\kappa^2 - \kappa'^2)\kappa^4\kappa'^4.\frac{1}{2}\pi$

$[6, 8] = [8, 6] = \frac{1}{7}\kappa^6\kappa'^6.\frac{1}{2}\pi$

These are the only functions of this kind which are needed for the evaluations of integrals in this paper.

* See for example DURÈGE'S 'Theorie der Elliptischen Functionen,' p. 293.

When these functions are introduced into (1) and the terms re-arranged, I find :—

$$I_i^s(\cos) \div \frac{4\pi k^3 \cos\beta\cos\gamma}{\sin^3\beta} =$$

$$A_0^2 - \tfrac{1}{3}\kappa^2\kappa'^2 A_1^2 + \tfrac{1}{5}\kappa^4\kappa'^4 A_2^2 - \tfrac{1}{7}\kappa^6\kappa'^6 A_3^2 + \tfrac{1}{9}\kappa^8\kappa'^8 A_4^2 - \ldots$$

$$+ 2(\kappa^2 - \kappa'^2)\left[\tfrac{1}{1.3} A_0A_1 - \tfrac{2}{3.5}\kappa^2\kappa'^2 A_1A_2 + \tfrac{3}{5.7}\kappa^4\kappa'^4 A_2A_3 - \tfrac{4}{7.9}\kappa^6\kappa'^6 A_3A_4 + \ldots\right]$$

$$+ \tfrac{2}{1.3.5}\left[\tfrac{1}{2}(2.4) - 3^2\kappa^2\kappa'^2\right] A_0A_2 - \tfrac{2}{3.5.7}\left[\tfrac{1}{2}(4.6) - 5^2\kappa^2\kappa'^2\right]\kappa^2\kappa'^2 A_1A_3$$

$$+ \tfrac{2}{5.7.9}\left[\tfrac{1}{2}(6.8) - 7^2\kappa^2\kappa'^2\right]\kappa^4\kappa'^4 A_2A_4 - \tfrac{2}{7.9.11}\left[\tfrac{1}{2}(8.10) - 9^2\kappa^2\kappa'^2\right]\kappa^6\kappa'^6 A_3A_5 + \ldots$$

$$+ 2(\kappa^2 - \kappa'^2)\left[\tfrac{2}{1.3.5.7}(3.5(1-\kappa^2\kappa'^2) - 3)A_0A_3 - \tfrac{3}{3.5.7.9}(5.7(1-\kappa^2\kappa'^2) - 3)\kappa^2\kappa'^2 A_1A_4\right.$$

$$\left. + \tfrac{4}{5.7.9.11}(7.9(1-\kappa^2\kappa'^2) - 3)\kappa^4\kappa'^4 A_2A_5 - \ldots\right]$$

$$+ \tfrac{2}{1.3.5.7.9}(192 - 816\kappa^2\kappa'^2 + 525\kappa^4\kappa'^4)A_0A_4 - \ldots \quad (2).$$

In this result a good many terms are added which are not deducible from the table of functions given above, but every term as stated here has actually been computed. The laws governing the succession of terms in the first six lines seem clear, but I do not claim that the proof of the laws is rigorous. I do not perceive how each series is derived from those preceding it, and I have no idea how the series beginning with A_0A_4 would go on. With sufficient patience it would no doubt be possible to determine the general law of the series, but I do not propose to make the attempt at present, since we have more than enough for the immediate object in view.

This result (2) is, of course, equally applicable to the integrals of the type I_i^s (sin).

In order to effect the required integrations we must define the functions, and I take the definitions (with a few very slight changes) from § 2 of "the Pear-shaped Figure." In order to use the preceding analysis it is necessary that the square of the P-function and the square of the cosine or sine function should be the same functions of $\kappa^2\cos^2\theta$ and of $-\kappa'^2\sin^2\phi$. But as in the definitions to be used this symmetry does not hold good, a difficulty arises, which may, however, be easily overcome. If the P-function be multiplied by any factor f, and the cosine or sine function by any factor g, the integral will be multiplied by f^2g^2. I therefore introduce such factors f and g as will render the residual factors of the squares of the P and cosine or sine-functions symmetrical in the proper manner.

It seems desirable to show how the results found here accord with the approximate integrals as found on pp. 548-9, § 22, of "Harmonics." In this connection I remark that $\frac{k^3\cos\beta\cos\gamma}{\sin^3\beta}$, when written in the notation of "Harmonics," is $k^3\nu(\nu^2-1)^{\frac{1}{2}}\left(\nu^2 - \frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$, a factor which I denoted in that paper by M.

It does not seem necessary to give full details of the analysis in the several cases, since it is sometimes tedious, and it merely involves the substitution in the formula of the values of A_0, A_1, A_2, &c.

We will now take the several harmonics successively.

Harmonic of the Order Zero.

This harmonic is simply unity, so that $A_0 = 1$ and all other A's vanish. The formula is

$$I_0 (\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{\sin^3\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

This is obviously right since the integral is $\int p d\sigma$, of which this is the known value.

Harmonics of the First Order.

Here we have all the A's zero excepting A_0 and A_1, and when the functions have the proper symmetrical forms, we have from (2),

$$I_i^s\binom{\cos}{\sin} = \frac{4\pi k^3 \cos\beta\cos\gamma}{\sin^3\beta}\left[A_0^2 - \tfrac{1}{3}\kappa^2\kappa'^2 A_1^2 + \tfrac{2}{3}(\kappa^2 - \kappa'^2) A_0 A_1\right], \; (s = 0, 1).$$

(1) *The Zonal Harmonic.*

I define this thus,

$$\left.\begin{aligned} \mathfrak{P}_1(\mu) &= \sin\theta = f.(\kappa^2 - \kappa^2\cos^2\theta)^{\frac{1}{2}}, \\ \mathfrak{C}_1(\phi) &= (1 - \kappa'^2\cos^2\phi)^{\frac{1}{2}} = g.(\kappa^2 + \kappa'^2\sin^2\phi)^{\frac{1}{2}}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4).$$

where $f = \dfrac{1}{\kappa}$, $g = 1$.

On squaring $\mathfrak{P}_1(\mu)$, it is clear that

$$A_0 = \kappa^2, \qquad A_1 = -1.$$

Whence I find

$$I_1(\cos) = \frac{4\pi k^3 \cos\beta\cos\gamma}{3\sin^3\beta} . f^2 g^2 \kappa^2.$$

Since with definition (4) $f^2 g^2 \kappa^2 = 1$,

$$I_1(\cos) = \frac{4\pi k^3 \cos\beta\cos\gamma}{3\sin^3\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

In "Harmonics" this harmonic is defined by

$$\mathfrak{P}_1(\mu) = P_1(\mu) = \mu\,; \qquad \mathfrak{C}_1(\phi) = \sqrt{(1 - \beta\cos 2\phi)} \quad . \quad . \quad . \quad . \quad (6).$$

Now we must take for f and g values such as to bring the two definitions into accord. This is the case if

$$f = \frac{1}{\kappa} = \sqrt{\frac{1+\beta}{1-\beta}}, \qquad g = \sqrt{1+\beta};$$

and $f^2 g^2 \kappa^2 = 1 + \beta$.

Hence

$$I_1(\cos) = \frac{4\pi M}{3}(1+\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

agreeing with the result on p. 549 of "Harmonics" for the case $i = 1$, $s = 0$, type OEC.

(2) *The Sectorial Cosine Harmonic.*

I define this thus,

$$\left.\begin{aligned} \mathsf{P}_1{}^1(\mu) &= (1 - \kappa^2 \sin^2\theta)^{\frac{1}{2}} = f\,.\,(\kappa'^2 + \kappa^2\cos^2\theta)^{\frac{1}{2}}, \\ \mathfrak{C}_1{}^1(\phi) &= \cos\phi \qquad\qquad\quad\;\; = g\,.\,(\kappa'^2 - \kappa'^2\sin^2\phi)^{\frac{1}{2}}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8),$$

where $f = 1$, $g = \dfrac{1}{\kappa'}$.

By symmetry with the last result

$$I_1{}^1(\cos) = \frac{4\pi k^3 \cos\beta\cos\gamma}{3\sin^3\beta}\,.\,f^2 g^2 \kappa'^2 = \frac{4\pi k^3\cos\beta\cos\gamma}{3\sin^3\beta} \quad . \quad . \quad . \quad . \quad . \quad (9).$$

In "Harmonics" I defined the functions thus,

$$\left.\begin{aligned} \mathsf{P}_1{}^1(\mu) &= P_1{}^1(\mu)\left(\frac{\frac{1+\beta}{1-\beta} - \mu^2}{1-\mu^2}\right)^{\frac{1}{2}} = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\left(1 - \frac{1-\beta}{1+\beta}\sin^2\theta\right)^{\frac{1}{2}}, \\ \mathfrak{C}_1{}^1(\phi) &= \cos\phi \end{aligned}\right\} \quad . \quad (10).$$

If we take $f = \left(\dfrac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}$, $g\kappa' = 1$, the two definitions agree, and we have

$$I_1{}^1(\cos) = \tfrac{4}{3}\pi M\,\frac{1+\beta}{1-\beta} = \tfrac{4}{3}\pi M(1 + 2\beta + 2\beta^2) \quad . \quad . \quad . \quad . \quad . \quad (11).$$

This agrees with the result on p. 549 of "Harmonics" with $i = 1$, $s = 1$, type OOC.

(3) *The Sectorial Sine Harmonic.*

I define this thus,

$$\left.\begin{aligned} \mathfrak{P}_1{}^1(\mu) &= \cos\theta = f\,.\,\kappa\cos\theta, \\ \mathfrak{S}_1{}^1(\phi) &= \sin\phi = g\,.\,\sqrt{-1}\,.\,\kappa'\sin\phi, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

where $f = \dfrac{1}{\kappa}$, $g = \dfrac{1}{\kappa'\sqrt{-1}}$.

On squaring $\mathfrak{P}_1{}^1$ we find $A_0 = 0$, $A_1 = 1$, and

$$I_1{}^1(\sin) = \frac{4\pi k^3 \cos\beta\cos\gamma}{3\sin^3\beta}(-f^2g^2\kappa^2\kappa'^2) = \frac{4\pi k^3 \cos\beta\cos\gamma}{3\sin^3\beta} \quad . \quad . \quad (13)$$

In "Harmonics" the definitions were the same, and therefore

$$I_1{}^1(\sin) = \frac{4\pi \mathrm{M}}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

This agrees with the result on p. 548 of "Harmonics" with $i = 1$, $s = 1$, type OOS.

Harmonics of the Second Order.

In these the only coefficients are A_0, A_1, A_2, and (2) becomes,

$$I_2{}^s\binom{\cos}{\sin} = \frac{4\pi k^3\cos\beta\cos\gamma}{\sin^3\beta}\left[A_0{}^2 - \tfrac{1}{3}\kappa^2\kappa'^2 A_1{}^2 + \tfrac{1}{5}\kappa^4\kappa'^4 A_2{}^2\right.$$
$$\left. + \tfrac{2}{3}(\kappa^2 - \kappa'^2)A_0A_1 - \tfrac{4}{15}(\kappa^2-\kappa'^2)\kappa^2\kappa'^2 A_1A_2 + \tfrac{2}{15}(4 - 9\kappa^2\kappa'^2)A_0A_2\right],$$

with $s = 0, 1, 2$.

(1) *and* (4) *The Zonal and Sectorial Cosine Harmonics.*

These are defined thus,

$$\left.\begin{aligned}\mathfrak{P}_2{}^s(\mu) &= \kappa^2\sin^2\theta - q^2,\\ \mathfrak{C}_2{}^s(\phi) &= q'^2 - \kappa'^2\cos^2\phi, \quad (s = 0, 2)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where $q^2 = \frac{1}{3}[1 + \kappa^2 \mp (1 - \kappa^2\kappa'^2)^{\frac{1}{2}}]$, with upper sign for $s = 0$ and lower for $s = 2$; and $q'^2 = 1 - q^2$.

Writing

$$t^2 = \kappa^2 - q^2 = q'^2 - \kappa'^2 = \tfrac{1}{3}[\kappa^2 - \kappa'^2 \pm (1 - \kappa^2\kappa'^2)^{\frac{1}{2}}],$$

$$\left.\begin{aligned}\mathfrak{P}_2{}^s(\mu) &= f\,.\,(t^2 - \kappa^2\cos^2\theta),\\ \mathfrak{C}_2{}^s(\phi) &= g\,.\,(t^2 + \kappa'^2\sin^2\phi), \quad (s = 0, 2)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where $f = 1$, $g = 1$. It may be noted that t^2 is a symmetrical function in κ^2 and $-\kappa'^2$.

Squaring $\mathfrak{P}_2{}^s$ we find

$$A_0 = t^4, \quad A_1 = -2t^2, \quad A_2 = 1,$$

After reduction I find, for $s = 0, 2$,

$$I_i^s(\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{3 \sin^3\beta}\left[t^8 - \tfrac{4}{3}(\kappa^2 - \kappa'^2)t^6 + \tfrac{2}{15}(4 - 19\kappa^2\kappa'^2)t^4 + \tfrac{8}{15}(\kappa^2 - \kappa'^2)\kappa^2\kappa'^2 t^2 + \tfrac{1}{5}\kappa^4\kappa'^4\right].$$

Now

$$3t^2 = \kappa^2 - \kappa'^2 \pm (1 - \kappa^2\kappa'^2)^{\frac{1}{2}},$$

$$9t^4 = 2 - 5\kappa^2\kappa'^2 \pm 2(\kappa^2 - \kappa'^2)(1 - \kappa^2\kappa'^2)^{\frac{1}{2}},$$

$$27t^6 = (4 - 7\kappa^2\kappa'^2)(\kappa^2 - \kappa'^2) \pm (4 - 13\kappa^2\kappa'^2)(1 - \kappa^2\kappa'^2)^{\frac{1}{2}},$$

$$81t^8 = 8 - 40\kappa^2\kappa'^2 + 41\kappa^4\kappa'^4 \pm 4(2 - 5\kappa^2\kappa'^2)(\kappa^2 - \kappa'^2)(1 - \kappa^2\kappa'^2)^{\frac{1}{2}}.$$

Whence on substitution, with $f^2g^2 = 1$,

$$I_2^s(\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{5 \sin^3\beta} \cdot \frac{2^3}{3^4}\left[(1 - \kappa^2\kappa'^2)^2 \pm (1 + \tfrac{1}{2}\kappa^2\kappa'^2)(\kappa^2 - \kappa'^2)(1 - \kappa^2\kappa'^2)^{\frac{1}{2}}\right] \quad . \quad (16).$$

The upper sign being taken for the zonal ($s = 0$), the lower for the sectorial harmonic ($s = 2$).

If these expressions be developed in powers of κ' as far as three terms of the series, I find, on re-introducing the factor f^2g^2,

$$I_2(\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{5 \sin^3\beta} \cdot \frac{2^4}{5^4}(1 - 2\kappa'^2 + \tfrac{21}{16}\kappa'^4) \cdot f^2g^2$$

$$= \frac{4\pi M}{5} \cdot \frac{2^4}{5^4}(1 - 4\beta + \tfrac{37}{4}\beta^2) \cdot f^2g^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

$$I_2^2(\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{5 \sin^3\beta} \cdot \tfrac{1}{3}\kappa'^4(1 - \kappa'^2 + \tfrac{11}{16}\kappa'^4) \cdot f^2g^2$$

$$= \frac{4\pi M}{5} \cdot \tfrac{1}{3}\left(\frac{2\beta}{1 + \beta}\right)^2(1 - 2\beta + \tfrac{19}{4}\beta^2) \cdot f^2g^2 \quad . \quad . \quad . \quad . \quad (18).$$

In "Harmonics" I made the following definitions

$$\left.\begin{aligned} \mathfrak{P}_2(\mu) &= P_2(\mu) - \tfrac{1}{4}\beta P_2^2(\mu) = 1 - \tfrac{3}{2}(1 + \tfrac{1}{2}\beta)\cos^2\theta, \\ \mathfrak{C}_2(\phi) &= 1 - \tfrac{3}{2}\beta\cos 2\phi \qquad = 1 - \tfrac{3}{2}\beta + \tfrac{3}{2}\beta\sin^2\phi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (19).$$

In order to make the two forms of definition agree we must take

$$ft^2 = 1, \quad gt^2 = 1 - \tfrac{3}{2}\beta.$$

Thus

$$f^2g^2 = \frac{1}{t^8}(1 - 3\beta + \tfrac{9}{4}\beta^2).$$

Now on development

$$t^8 = (\tfrac{2}{3})^4(1 - 5\kappa'^2 + \tfrac{81}{8}\kappa'^4) = (\tfrac{2}{3})^4(1 - 10\beta + \tfrac{101}{2}\beta^2).$$

Whence

$$f^2g^2 = (\tfrac{3}{2})^4(1 + 7\beta + \tfrac{87}{4}\beta^2).$$

Introducing this I find

$$I_2(\cos) = \frac{4\pi M}{5}(1 + 3\beta + 3\beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

agreeing with the result on p. 549 of "Harmonics" with $i = 2$, $s = 0$, type EEC.

Again in "Harmonics"

$$\left.\begin{aligned} \mathfrak{P}_2{}^2(\mu) &= 3\beta P_2(\mu) + P_2{}^2(\mu) = 3\beta + 3(1 - \tfrac{3}{2}\beta)\cos^2\theta, \\ \mathfrak{C}_2{}^2(\phi) &= \tfrac{1}{2}\beta + \cos 2\phi \qquad\quad = 1 + \tfrac{1}{2}\beta - 2\sin^2\phi \end{aligned}\right\} \quad . \quad . \quad . \quad (21).$$

In order to make the two definitions agree we must take

$$f\kappa^2 = -3(1 - \tfrac{3}{2}\beta), \qquad g\kappa'^2 = 2;$$

or

$$f = -3(1 + \tfrac{1}{2}\beta - \beta^2), \quad g = 2\left(\frac{1+\beta}{2\beta}\right).$$

So that $f^2g^2 = 2^2 . 3^2 \left(\frac{1+\beta}{2\beta}\right)^2 (1 + \beta - \tfrac{7}{4}\beta^2)$.

Introducing this in (18) we have

$$I_2{}^2(\cos) = \frac{4\pi}{7} M . 12(1 - \beta + \beta^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

agreeing with the result on p. 548 of "Harmonics" with $i = 2$, $s = 2$, type EEC.

(2) *The Cosine Tesseral Harmonic.*

These are defined thus,—

$$\left.\begin{aligned} \mathsf{P}_2{}^1(\mu) &= \sin\theta(1 - \kappa^2\sin^2\theta)^{\frac{1}{2}} = f . (\kappa^2 - \kappa^2\cos^2\theta)^{\frac{1}{2}}(\kappa'^2 + \kappa^2\cos^2\theta)^{\frac{1}{2}}, \\ \mathsf{C}_2{}^1(\phi) &= \cos\phi(1 - \kappa'^2\cos^2\phi)^{\frac{1}{2}} = g(\kappa^2 + \kappa'^2\sin^2\phi)^{\frac{1}{2}}(\kappa'^2 - \kappa'^2\sin^2\phi)^{\frac{1}{2}}, \end{aligned}\right\}. \quad (23)$$

where $f = \frac{1}{\kappa}$, $g = \frac{1}{\kappa'}$.

Squaring $\mathsf{P}_2{}^1$ we find

$$A_0 = \kappa^2\kappa'^2, \; A_1 = (\kappa^2 - \kappa'^2), \; A_2 = -1.$$

On substituting in the formula, I find, on putting $f^2g^2\kappa^2\kappa'^2 = 1$ and reducing,

$$I_2{}^1(\cos) = \frac{4\pi k^3\cos\beta\cos\gamma}{5\sin^3\beta} . \tfrac{1}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

In "Harmonics" the definitions were

$$\left.\begin{aligned}\mathsf{P}_2^{\,1}(\mu) &= \left(\frac{\frac{1+\beta}{1-\beta}-\mu^2}{1-\mu^2}\right)^{\frac{1}{2}} P_2^{\,1}(\mu) = 3\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\sin\theta\left(1-\frac{1-\beta}{1+\beta}\sin^2\theta\right)^{\frac{1}{2}}\\ \mathsf{C}_2^{\,1}(\phi) &= (1-\beta\cos 2\phi)^{\frac{1}{2}}\cos\phi = (1+\beta)^{\frac{1}{2}}\cos\phi\left(1-\frac{2\beta}{1+\beta}\cos^2\phi\right)^{\frac{1}{2}}\end{aligned}\right\} \quad (25).$$

In order to make the two definitions agree we must take

$$f\kappa = 3\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}, \qquad g\kappa' = (1+\beta)^{\frac{1}{2}};$$

so that $f^2g^2\kappa^2\kappa'^2 = 3^2\,\dfrac{(1+\beta)^2}{1-\beta} = 3^2\,(1+3\beta+4\beta^2)$. On multiplying (24) by this factor, we have

$$I_2^{\,1}(\cos) = \frac{4\pi}{5}\,\mathrm{M}\,.\,3\,(1+3\beta+4\beta^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

agreeing with the result on p. 549 of "Harmonics" with $i = 2$, $s = 1$, type EOC.

(3) *The Tesseral Sine Harmonic.*

This is defined thus,—

$$\left.\begin{aligned}\mathfrak{P}_2^{\,1}(\mu) &= \sin\theta\cos\theta = f\,.\,\kappa\cos\theta\,(\kappa^2-\kappa^2\cos^2\theta)^{\frac{1}{2}},\\ \mathsf{S}_2^{\,1}(\phi) &= \sin\phi\,(1-\kappa'^2\cos^2\phi)^{\frac{1}{2}} = g\,.\,\sqrt{-1}\,\kappa'\sin\phi\,(\kappa^2+\kappa'^2\sin^2\phi)^{\frac{1}{2}},\end{aligned}\right\}. \quad (27)$$

where $f = \dfrac{1}{\kappa^2}$, $g = \dfrac{1}{\kappa'\sqrt{-1}}$.

Squaring $\mathfrak{P}_2^{\,1}$ we find

$$A_0 = 0,\ A_1 = \kappa^2,\ A_2 = -1$$

whence, on putting $-f^2g^2\kappa^4\kappa'^2 = 1$,

$$I_2^{\,1}(\sin) = \frac{4\pi k^3\cos\beta\cos\gamma}{5\sin^3\beta}\,.\,\tfrac{1}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

In "Harmonics" the definitions were

$$\left.\begin{aligned}\mathfrak{P}_2^{\,1}(\mu) &= P_2^{\,1}(\mu) = 3\sin\theta\cos\theta,\\ \mathsf{S}_2^{\,1}(\phi) &= \sin\phi\,(1-\beta\cos 2\phi)^{\frac{1}{2}} = (1+\beta)^{\frac{1}{2}}\sin\phi\left(1-\frac{2\beta}{1+\beta}\cos^2\phi\right)^{\frac{1}{2}}\end{aligned}\right\} \quad (29).$$

Therefore, to make the two definitions agree, we must take

$$f\kappa^2 = 3, \qquad g\kappa'\sqrt{-1} = (1+\beta)^{\frac{1}{2}}.$$

Therefore $-f^2g^2\kappa^4\kappa'^2 = 3^2(1+\beta)$, and on multiplying (28) by this factor we have

$$I_2^1(\sin) = \frac{4\pi}{5}\mathrm{M}\,.\,3\,(1+\beta), \quad . \quad . \quad . \quad (30)$$

agreeing with the result on p. 549 of "Harmonics" with $i=2$, $s=1$, type EOS.

(5) *The Sectorial Sine Harmonic.*

This is defined thus:—

$$\left.\begin{aligned}\mathsf{P}_2^2(\mu) &= \cos\theta\,(1-\kappa^2\sin^2\theta)^{\frac{1}{2}},\\ \mathfrak{S}_2^2(\phi) &= \sin\phi\cos\phi.\end{aligned}\right\} \quad . \quad . \quad . \quad (31).$$

If in the last integral we had written $\frac{1}{2}\pi-\theta$ for ϕ, and $\frac{1}{2}\pi-\phi$ for θ, and κ' for κ, $\mathfrak{P}_2^1$ would have become $\mathfrak{S}_2^2$, and S_2^1 would have become P_2^2. Therefore the result (28) gives what is needed by merely interchanging κ and κ'.

Therefore

$$I_2^2(\sin) = \frac{4\pi k^3\cos\beta\cos\gamma}{5\sin^3\beta}\,.\,\tfrac{1}{3} \quad . \quad . \quad . \quad (32).$$

For the purpose of comparison I must put

$$\mathsf{P}_2^2(\mu) = f\,.\,\kappa\cos\theta\,(\kappa'^2+\kappa^2\cos^2\theta)^{\frac{1}{2}}, \quad \mathfrak{S}_2^2(\phi) = g\,.\,\sqrt{-1}\,\kappa'\sin\phi\,(\kappa'^2-\kappa'^2\sin^2\phi)^{\frac{1}{2}}, \quad (31)$$

and

$$I_2^2(\sin) = \frac{4\pi k^3\cos\beta\cos\gamma}{5\sin^3\beta}\left(-\tfrac{1}{3}f^2g^2\kappa^2\kappa'^4\right) \quad . \quad . \quad . \quad (33).$$

In "Harmonics" the definition was

$$\left.\begin{aligned}\mathsf{P}_2^2(\mu) &= \left(\frac{\frac{1+\beta}{1-\beta}-\mu^2}{1-\mu^2}\right)^{\frac{1}{2}} P_2^2(\mu) = 3\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\cos\theta\left(1-\frac{1-\beta}{1+\beta}\sin^2\theta\right)^{\frac{1}{2}},\\ \mathfrak{S}_2^2(\phi) &= \sin 2\phi \qquad\qquad = 2\sin\phi\cos\phi.\end{aligned}\right\} \,.\, (34).$$

In order to make the two definitions agree we must take

$$f\kappa = 3\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}, \qquad g\sqrt{-1}\,.\,\kappa'^2 = 2.$$

Thus $-f^2g^2\kappa^2\kappa'^4 = 2^2\,.\,3^2\,\dfrac{1+\beta}{1-\beta}$; introducing this in (33) we have

$$I_2^2(\sin) = \frac{4\pi\mathrm{M}}{5}\,.\,12\left(\frac{1+\beta}{1-\beta}\right) = \frac{4\pi\mathrm{M}}{5}\,12\,(1+2\beta+2\beta^2). \quad . \quad . \quad . \quad (35)$$

agreeing with the result on p. 548 of "Harmonics" with $i=2$, $s=2$, type EES.

The Harmonics of the Third Order.

In these the only coefficients are A_0, A_1, A_2, A_3, and (2) becomes

$$I_3^s\binom{\cos}{\sin} = \frac{4\pi k^3 \cos\beta\cos\gamma}{\sin^3\beta}\{A_0^2 - \tfrac{1}{3}\kappa^2\kappa'^2 A_1^2 + \tfrac{1}{5}\kappa^4\kappa'^4 A_2^2 - \tfrac{1}{7}\kappa^6\kappa'^6 A_3^2$$
$$+ 2(\kappa^2-\kappa'^2)[\tfrac{1}{3}A_0A_1 - \tfrac{2}{15}\kappa^2\kappa'^2 A_1A_2 + \tfrac{3}{35}\kappa^4\kappa'^4 A_2A_3]$$
$$+ \tfrac{2}{15}(4-9\kappa^2\kappa'^2)A_0A_2 - \tfrac{2}{105}(12-25\kappa^2\kappa'^2)\kappa^2\kappa'^2 A_1A_3$$
$$+ \tfrac{4}{35}(\kappa^2-\kappa'^2)(4-5\kappa^2\kappa'^2)A_0A_3\} \quad (s = 0, 1, 2, 3).$$

(1) *and* (4) *The Zonal and Second Tesseral Cosine Harmonics.*

These are defined thus :—

$$\left.\begin{aligned}\mathfrak{P}_3^s(\mu) &= \sin\theta(\kappa^2\sin^2\theta - q^2)\\ \mathfrak{C}_3^s(\phi) &= (q'^2 - \kappa'^2\cos^2\phi)(1-\kappa'^2\cos^2\phi)^{\frac{1}{2}}, \quad (s = 0, 2)\end{aligned}\right\} \quad . \quad . \quad . \quad (36)$$

where $q^2 = \frac{2}{5}[1+\kappa^2 \mp (1-\frac{7}{4}\kappa^2+\kappa^4)^{\frac{1}{2}}]$, with the upper sign for $s = 0$ and the lower for $s = 2$. Writing

$$t^2 = \kappa^2 - q^2 = q'^2 - \kappa'^2$$
$$= \tfrac{1}{5}[3\kappa^2 - 2 \pm (4 - 7\kappa^2 + 4\kappa^4)^{\frac{1}{2}}] = \tfrac{1}{5}[1 - 3\kappa'^2 \pm (1-\kappa'^2+4\kappa'^4)^{\frac{1}{2}}]$$

$$\left.\begin{aligned}\mathfrak{P}_3^s(\mu) &= f\,.\,(t^2 - \kappa^2\cos^2\theta)(\kappa^2-\kappa^2\cos^2\theta)^{\frac{1}{2}},\\ \mathfrak{C}_3^s(\phi) &= g\,.\,(t^2+\kappa'^2\sin^2\phi)(\kappa^2+\kappa'^2\sin^2\phi)^{\frac{1}{2}},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (37)$$

where $f = \dfrac{1}{\kappa}$, $g = 1$.

Squaring $\mathfrak{P}_3^s$ we find

$$A_0 = t^4\kappa^2, \quad A_1 = -(2\kappa^2+t^2)t^2, \quad A_2 = 2t^2+\kappa^2, \quad A_3 = -1.$$

After some rather tedious reductions I find (for $s = 0, 2$)

$$I_3^s(\cos) = \frac{4\pi k^3\cos\beta\cos\gamma}{\sin^3\beta}\kappa^2\{\tfrac{1}{3}t^8 - \tfrac{4}{15}(1-3\kappa'^2)t^6 + \tfrac{2}{105}(4-25\kappa'^2+33\kappa'^4)t^4$$
$$+ \tfrac{4}{105}(2-5\kappa'^2)\kappa^2\kappa'^2t^2 + \tfrac{1}{35}\kappa^4\kappa'^4\}f^2g^2.$$

Now writing $D = (1-\kappa'^2+4\kappa'^4)^{\frac{1}{2}}$,

$$5t^2 = 1 - 3\kappa'^2 \pm D,$$
$$5^2t^4 = 2 - 7\kappa'^2 + 13\kappa'^4 \pm 2(1-3\kappa'^2)D,$$
$$5^3t^6 = 4 - 21\kappa'^2 + 48\kappa'^4 - 63\kappa'^6 \pm (4 - 19\kappa'^2 + 31\kappa'^4)D,$$
$$5^4t^8 = 8 - 56\kappa'^2 + 177\kappa'^4 - 314\kappa'^6 + 313\kappa'^8 \pm (8 - 52\kappa'^2 + 136\kappa'^4 - 156\kappa'^6)D.$$

On substituting these in the above expression, and noting that $\kappa^2 f^2 g^2$ will be unity with the definition adopted, I find

$$\left.\begin{aligned} I_3(\cos) &= \frac{4\pi k^3 \cos\beta\cos\gamma}{7\sin^3\beta}\cdot\frac{2^3}{5^4} D[(1-\tfrac{1}{2}\kappa'^2)(1-\kappa'^2-\tfrac{8}{3}\kappa'^4)+(1-\kappa'^2+\tfrac{2}{3}\kappa'^4)D], \\ I_3^2(\cos) &= \text{the same with the sign of } D \text{ changed.} \end{aligned}\right\} \quad (38).$$

If these expressions be developed in powers of κ', and if the factor $\kappa^2 f^2 g^2$ be re-introduced, I find

$$I_3(\cos) = \frac{4\pi k^3\cos\beta\cos\gamma}{7\sin^3\beta}(\tfrac{2}{5})^4(1-2\kappa'^2+\tfrac{49}{16}\kappa'^4)\,.\,\kappa^2 f^2 g^2.$$

$$= \frac{4\pi}{7}\,\mathrm{M}\,(\tfrac{2}{5})^4(1-4\beta+\tfrac{65}{4}\beta^2)\,.\,\kappa^2 f^2 g^2$$

$$I_3^2(\cos) = \frac{4\pi k^3\cos\beta\cos\gamma}{7\sin^3\beta}\cdot\tfrac{1}{3\cdot 5}\kappa'^4(1-\kappa'^2+\tfrac{31}{16}\kappa'^4)\,.\,\kappa^2 f^2 g^2$$

$$= \frac{4\pi}{7}\,\mathrm{M}\,.\,\tfrac{1}{3\cdot 5}\kappa'^4(1-2\beta+\tfrac{39}{4}\beta^2)\,.\,\kappa^2 f^2 g^2.$$

In "Harmonics" I defined,

$$\mathfrak{P}_3(\mu) = P_3(\mu) - \tfrac{1}{4}\beta P_3^2(\mu) = \sin\theta\,[\tfrac{5}{2}\sin^2\theta(1+\tfrac{3}{2}\beta)-\tfrac{3}{2}(1+\tfrac{5}{2}\beta)]. \quad (39).$$

In order that our previous definition may agree with this we must have

$$f\kappa q^2 = \tfrac{3}{2}(1+\tfrac{5}{2}\beta), \quad f\kappa^3 = \tfrac{5}{2}(1+\tfrac{3}{2}\beta).$$

Now
$$\begin{aligned} q^2 &= \tfrac{1}{5}[4-2\kappa'^2-\sqrt{(1-\kappa'^2+4\kappa'^4)}] = \tfrac{3}{5}[1-\tfrac{1}{2}\kappa'^2-\tfrac{5}{8}\kappa'^4] \\ &= \tfrac{3}{5}[1-\beta-\tfrac{3}{2}\beta^2] \end{aligned}$$

whence $f\kappa = \frac{5}{2}(1+\frac{7}{2}\beta+5\beta^2)$, and this value of $f\kappa$ satisfies the second equation.

With regard to $\mathsf{C}_3(\phi)$ there is a mistake in the table (the only one I have detected therein) on p. 556 of "Harmonics," for the coefficient of the second term should not be 3β but $\frac{5}{2}\beta$. The mistake obviously arose from my using the formula for p_2 instead of that for p'_2 as given on p. 490.

With the corrected coefficient the definition is

$$\left.\begin{aligned} \mathsf{C}_3(\phi) &= (1-\beta\cos 2\phi)^{\frac{1}{2}}(1-\tfrac{5}{2}\beta\cos 2\phi) \\ &= (1+\beta)^{\frac{1}{2}}(1+\tfrac{5}{2}\beta-5\beta\cos^2\phi)\left(1-\frac{2\beta}{1+\beta}\cos^2\phi\right)^{\frac{1}{2}} \end{aligned}\right\} \quad (40).$$

In order that the previous definition may agree with this we must have

$$g q'^2 = (1+\beta)^{\frac{1}{2}}(1+\tfrac{5}{2}\beta) = 1+3\beta+\tfrac{9}{8}\beta^2,$$

$$g\kappa'^2 = 5\beta(1+\beta)^{\frac{1}{2}}.$$

But $q'^2 = 1 - q^2 = \frac{2}{5}(1 + \frac{3}{2}\beta + \frac{9}{4}\beta^2)$, and thence

$$g = \tfrac{5}{2}(1 + \tfrac{3}{2}\beta - \tfrac{27}{8}\beta^2).$$

This value of g will be found to give the correct value for $g\kappa'^2$.

Then $$fg\kappa = (\tfrac{5}{2})^2(1 + 5\beta + \tfrac{55}{8}\beta^2),$$

and $$f^2g^2\kappa^2 = (\tfrac{5}{2})^4(1 + 10\beta + \tfrac{155}{4}\beta^2).$$

Introducing this into the value of I_3 (cos), we find

$$I_3(\cos) = \frac{4\pi}{7}\mathrm{M}(1 + 6\beta + 15\beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

agreeing with the result on p. 549 of "Harmonics" for $i = 3$, $s = 0$, type OEC.

Again in "Harmonics" I defined

$$\mathfrak{P}_3^2(\mu) = 15\beta P_3(\mu) + P_3^2(\mu) = 15\sin\theta[1 - \tfrac{3}{2}\beta - (1 - \tfrac{5}{2}\beta)\sin^2\theta] \quad . \quad (42).$$

To make the former definition agree with this we must take

$$f\kappa q^2 = -15(1 - \tfrac{3}{2}\beta), \quad f\kappa^3 = -15(1 - \tfrac{5}{2}\beta).$$

In the present case

$$q^2 = \tfrac{1}{5}[4 - 2\kappa'^2 + \sqrt{(1 - \kappa'^2 + 4\kappa'^4)}] = 1 - \tfrac{1}{2}\kappa'^2 + \tfrac{3}{8}\kappa'^4 + \tfrac{3}{16}\kappa'^6,$$
$$= 1 - \beta + \tfrac{5}{2}\beta^2 - \tfrac{5}{2}\beta^3.$$

Omitting the term in β^3 we find, with this value of q^2,

$f\kappa = -15(1 - \frac{1}{2}\beta - 3\beta^2)$, and that the second equation is satisfied.

Again I defined

$$\mathfrak{C}_3^2(\phi) = (\tfrac{3}{2}\beta + \cos 2\phi)(1 - \beta\cos 2\phi)^{\frac{1}{2}} = (1+\beta)^{\frac{1}{2}}(2\cos^2\phi - 1 + \tfrac{3}{2}\beta)\left(1 - \frac{2\beta}{1+\beta}\cos^2\phi\right)^{\frac{1}{2}} (43).$$

Hence to secure agreement we must take

$$gq'^2 = -(1 - \tfrac{3}{2}\beta)(1 + \beta)^{\frac{1}{2}} = -(1 - \beta - \tfrac{7}{8}\beta^2),$$
$$g\kappa'^2 = -2(1 + \beta)^{\frac{1}{2}}.$$

Now $q'^2 = 1 - q^2 = \beta(1 - \frac{5}{2}\beta + \frac{5}{2}\beta^2)$, and therefore

$$g = -\frac{1}{\beta}(1 + \tfrac{3}{2}\beta + \tfrac{3}{8}\beta^2).$$

The second equation is satisfied.

We have then

$$f^2g^2\kappa^2\kappa'^4 = 2^2 \,.\, 3^2 \,.\, 5^2 (1 + \beta)(1 - \beta - \tfrac{23}{4}\beta^2) = 2^2 \,.\, 3^2 \,.\, 5^2 (1 - 0 \,.\, \beta - \tfrac{27}{4}\beta^2).$$

Introducing this into the value of $I_3^{\,2}(\cos)$, we find

$$I_3^{\,2}(\cos) = \frac{4\pi \mathrm{M}}{7} \,.\, 3 \,.\, 4 \,.\, 5 (1 - 2\beta + 3\beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (44),$$

agreeing with the result in p. 548 of "Harmonics" for $i = 3$, $s = 2$, type OEC.

(2) *and* (6) *First Tesseral Cosine Harmonic and Sectorial Cosine Harmonic.*

These are defined thus:—

$$\left.\begin{aligned} \mathsf{P}_3^{\,s}(\mu) &= (\kappa^2 \sin^2\theta - q^2)(1 - \kappa^2 \sin^2\theta)^{\frac{1}{2}}, \\ \mathfrak{C}_3^{\,s}(\phi) &= \cos\phi\,(q'^2 - \kappa'^2 \cos^2\phi), \quad (s = 1,\, 3) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (45),$$

where $q^2 = \frac{1}{5}(1 + 2\kappa^2 \mp (1 - \kappa^2 + 4\kappa^4)^{\frac{1}{2}})$, with upper sign for $s = 1$ and lower sign for $s = 3$, and $q'^2 = 1 - q^2$.

Writing $t'^2 = \kappa'^2 - q'^2$,

$$\left.\begin{aligned} \mathsf{P}_3^{\,s}(\mu) &= f(t'^2 + \kappa^2\cos^2\theta)(\kappa'^2 + \kappa^2\cos^2\theta)^{\frac{1}{2}}, \\ \mathfrak{C}_3^{\,s}(\phi) &= g\,(t'^2 - \kappa'^2\sin^2\phi)(\kappa'^2 - \kappa'^2\sin^2\phi)^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (46),$$

where $f = -1$, $g = -\dfrac{1}{\kappa'}$.

It is clear that $[\mathsf{P}_3^{\,s}(\mu)\mathfrak{C}_3^{\,s}(\phi)]^2 (s = 1, 3)$ has the same form as $[\mathfrak{P}_3^{\,s}(\mu)\mathsf{C}_3^{\,s}(\phi)]^2 (s = 0, 2)$ when in the latter we interchange θ with $\frac{1}{2}\pi - \phi$, and κ with κ'. The interchange of the variables of integration clearly makes no difference in the result, and therefore we need only interchange κ and κ', and replace t by t'.

In the present instance

$$t'^2 = \kappa'^2 - q'^2 = q^2 - \kappa^2 = \tfrac{1}{5}[1 - 3\kappa^2 \mp (1 - \kappa^2 + 4\kappa^4)^{\frac{1}{2}}].$$

This shows that t'^2 is the same function of κ^2 that t^2 was of κ'^2, but that $I_3^{\,1}(\cos)$ is analogous with $I_3^{\,2}(\cos)$, and $I_3^{\,3}(\cos)$ with $I_3(\cos)$. Thus we may at once write down the results by interchanging κ and κ' throughout.

Let $$D' = (1 - \kappa^2 + 4\kappa^4)^{\frac{1}{2}}.$$

Then putting $\kappa'^2 f^2 g^2 = 1$, we have by symmetry with (38)

$$\left.\begin{aligned} I_3^{\,1}(\cos) &= \frac{4\pi k^3 \cos\beta\cos\gamma}{7\sin^3\beta} \,.\, \frac{2^3}{5^4} D'[(1 - \kappa^2 + \tfrac{2}{3}\kappa^4)D' - (1 - \tfrac{1}{2}\kappa^2)(1 - \kappa^2 - \tfrac{8}{3}\kappa^4)], \\ I_3^{\,3}(\cos) &= \text{the same with the sign of } D' \text{ changed.} \end{aligned}\right\} (47).$$

If these expressions be developed in powers of κ' I find, on reintroducing the factor $\kappa'^2 f^2 g^2$,

$$I_3^1(\cos) = \frac{4\pi k^3 \cos\beta\cos\gamma}{7\sin^3\beta} \cdot \frac{2^7}{3.5^4} \cdot (1 - \tfrac{9}{4}\kappa'^2 + \tfrac{361}{256}\kappa'^4) \cdot \kappa'^2 f^2 g^2,$$

$$= \frac{4\pi}{7} \text{M} \cdot \frac{2^7}{3.5^4}(1 - \tfrac{9}{2}\beta + \tfrac{649}{64}\beta^2) \cdot \kappa'^2 f^2 g^2,$$

$$I_3^3(\cos) = \frac{4\pi k^3 \cos\beta\cos\gamma}{7\sin^3\beta} \cdot \tfrac{1}{2.5}\kappa'^4 (1 - \tfrac{3}{2}\kappa'^2 + \tfrac{183}{256}\kappa'^4) \cdot \kappa'^2 f^2 g^2,$$

$$= \frac{4\pi}{7} \text{M} \cdot \tfrac{1}{2.5}\left(\frac{2\beta}{1+\beta}\right)^2 (1 - 3\beta + \tfrac{375}{64}\beta^2) \cdot \kappa'^2 f^2 g^2.$$

In "Harmonics" I defined

$$\mathsf{P}_3^1(\mu) = \left(\frac{\frac{1+\beta}{1-\beta} - \mu^2}{1-\mu^2}\right)^{\frac{1}{2}} [P_3^1(\mu) - \tfrac{3}{16}\beta(1 + \tfrac{3}{4}\beta)\, P_3^3(\mu)],$$

$$= \tfrac{15}{2}\left(1 - \frac{1-\beta}{1+\beta}\sin^2\theta\right)^{\frac{1}{2}} \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} [\sin^2\theta\,(1 + \tfrac{3}{8}\beta + \tfrac{9}{32}\beta^2) - \tfrac{1}{5}(1 + \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2)] \quad (48).$$

But we have defined it above by

$$\mathsf{P}_3^1(\mu) = f(1 - \kappa^2\sin^2\theta)^{\frac{1}{2}}(\kappa^2\sin^2\theta - q^2).$$

Therefore
$$f\kappa^2 = \tfrac{15}{2}\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}(1 + \tfrac{3}{8}\beta + \tfrac{9}{32}\beta^2),$$

$$fq^2 = \tfrac{3}{2}\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}(1 + \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2).$$

Now $q^2 = \tfrac{1}{5}(1 - \tfrac{1}{4}\kappa'^2 - \tfrac{15}{64}\kappa'^4) = \tfrac{1}{5}(1 - \tfrac{1}{2}\beta - \tfrac{7}{16}\beta^2)$.

Whence $f = \tfrac{15}{2}(1 + \tfrac{27}{8}\beta + \tfrac{189}{32}\beta^2)$.

This value also satisfies the expression for $f\kappa^2$.

Again I defined

$$\mathfrak{C}_3^1(\phi) = \cos\phi - \tfrac{5}{8}\beta(1 + \tfrac{3}{4}\beta)\cos 3\phi,$$

$$= \cos\phi\,[1 + \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2 - \tfrac{5}{2}\beta(1 + \tfrac{3}{4}\beta)\cos^2\phi] \quad . \quad . \quad . \quad (49).$$

But we have defined it above by

$$\mathfrak{C}_3^1 = g\kappa'\cos\phi\,(\kappa'^2\cos^2\phi - q'^2).$$

Therefore

$$g\kappa' q'^2 = -(1 + \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2), \qquad g\kappa'^3 = -\tfrac{5}{2}\beta(1 + \tfrac{3}{4}\beta^2).$$

With the above value for q^2 we have $q'^2 = \tfrac{4}{5}(1 + \tfrac{1}{8}\beta + \tfrac{7}{64}\beta^2)$; whence

$$g\kappa' = -\tfrac{5}{4}(1 + \tfrac{7}{4}\beta + \tfrac{69}{64}\beta^2).$$

Therefore $f^2 g^2 \kappa'^2 = \frac{3^2 . 5^4}{2^6} [1 + \frac{41}{4}\beta + \frac{3331}{64}\beta^2]$, and I find

$$I_3^1 (\cos) = \frac{4\pi}{7} \text{M} . 6 (1 + \tfrac{23}{4}\beta + \tfrac{257}{16}\beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

agreeing with the result on p. 549 of "Harmonics" with $i = 3$, $s = 1$, type OOC.

In "Harmonics" I defined

$$\mathsf{P}_3^3 (\mu) = \left(\frac{\frac{1+\beta}{1-\beta} - \mu^2}{1 - \mu^2}\right)^{\frac{1}{2}} [\tfrac{5}{4}\beta (1 + \tfrac{3}{4}\beta) P_3^1 (\mu) + P_3^3 (\mu)],$$

$$= 15 \left(1 - \frac{1-\beta}{1+\beta}\sin^2\theta\right)^{\frac{1}{2}} [1 + \tfrac{7}{8}\beta + \tfrac{9}{32}\beta^2 - \sin^2\theta (1 + \tfrac{3}{8}\beta - \tfrac{19}{32}\beta^2)] \quad . \quad (51).$$

But $\mathsf{P}_3^3 (\mu) = f (1 - \kappa^2 \sin^2\theta)^{\frac{1}{2}} (\kappa^2 \sin^2\theta - q^2)$.

Therefore $f\kappa^2 = - 15 (1 + \frac{3}{8}\beta - \frac{19}{32}\beta^2)$, $\quad fq^2 = - 15 (1 + \frac{7}{8}\beta + \frac{9}{32}\beta^2)$.

Now $\qquad q^2 = 1 - \frac{3}{2^2}\kappa'^2 + \frac{3}{2^6}\kappa'^4 + \frac{21}{2^9}\kappa'^6 \ldots$

Therefore $\qquad q^2 = 1 - \frac{3}{2}\beta + \frac{27}{16}\beta^2$, and

$$f = - 15 (1 + \tfrac{19}{8}\beta + \tfrac{69}{32}\beta^2).$$

This also gives the correct value to $f\kappa^2$.

Again $\qquad \mathfrak{C}_3^3 (\phi) = \frac{3}{8}\beta (1 + \frac{3}{4}\beta) \cos\phi + \cos 3\phi$,

$$= \cos\phi [4 \cos^2\phi - 3 (1 - \tfrac{1}{8}\beta - \tfrac{3}{32}\beta^2)] \quad . \quad . \quad . \quad . \quad (52).$$

But $\mathfrak{C}_3^3 (\phi) = g\kappa' \cos\phi (\kappa'^2 \cos^2\phi - q'^2)$.

Therefore $g\kappa' = \frac{4}{\kappa'^2}$, and $g\kappa' . q'^2 = 3 (1 - \frac{1}{8}\beta - \frac{3}{32}\beta^2)$.

If we eliminate $g\kappa'$, these equations give the correct value for q'^2.

Then $f g\kappa' = - \frac{60}{\kappa'^2} (1 + \frac{19}{8}\beta + \frac{69}{32}\beta^2)$.

Therefore

$$f^2 g^2 \kappa'^2 = \frac{3^2 . 4^2 . 5^2}{\kappa'^4} (1 + \tfrac{19}{4}\beta + \tfrac{637}{64}\beta^2).$$

Hence we find

$$I_3^3 (\cos) = \frac{4\pi \text{M}}{7} . 360 (1 + \tfrac{7}{4}\beta + \tfrac{25}{16}\beta^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (53),$$

agreeing with the result on p. 548 of "Harmonics" with $i = 3$, $s = 3$, type OOC.

(3) *and* (7) *First Tesseral Sine Harmonic and Sectorial Sine Harmonic.*

These are defined thus :—

$$\left.\begin{aligned}\mathfrak{P}_3^s(\mu) &= \cos\theta\,(\kappa^2\sin^2\theta - q^2),\\ \mathfrak{S}_3^s(\phi) &= \sin\phi\,(q'^2 - \kappa'^2\cos^2\phi), \quad (s = 1, 3),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

where $q^2 = \frac{1}{5}[2 + \kappa^2 \mp (4 - \kappa^2\kappa'^2)^{\frac{1}{2}}]$, with the upper sign for $s = 1$, the lower for $s = 3$, and $q'^2 = 1 - q^2$.

Writing $\qquad t^2 = \kappa^2 - q^2,$

$$\left.\begin{aligned}\mathfrak{P}_3^s(\mu) &= f\,.\,\kappa\cos\theta\,(t^2 - \kappa^2\cos^2\theta),\\ \mathfrak{S}_3^s(\phi) &= g\,.\,\kappa'\sqrt{-1}\,\sin\phi\,(t^2 + \kappa'^2\sin^2\phi),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

where $f = \dfrac{1}{\kappa}, \quad g = \dfrac{1}{\kappa'\sqrt{-1}}$.

Squaring $\mathfrak{P}_3^s$ we find,

$$A_0 = 0, \quad A_1 = t^4, \quad A_2 = -2t^2, \quad A_3 = 1.$$

On substitution in the formula for harmonics of the third order I find

$$I_3^s(\sin) = \frac{4\pi k^3\cos\beta\cos\gamma}{\sin^3\beta}\left[\tfrac{1}{3}t^8 - \tfrac{8}{15}(\kappa^2 - \kappa'^2)\,t^6 + \tfrac{2}{105}(12 - 67\kappa^2\kappa'^2)\,t^4 + \tfrac{12}{35}(\kappa^2 - \kappa'^2)\,\kappa^2\kappa'^2 t^2 + \tfrac{1}{7}\kappa^4\kappa'^4\right](-f^2g^2\kappa^2\kappa'^2).$$

If we write

$$D = (1 - \tfrac{1}{4}\kappa^2\kappa'^2)^{\frac{1}{2}},$$

$$\frac{5}{2}\,t^2 = 1 - 2\kappa'^2 \pm D,$$

$$\frac{5^2}{2^2}\,t^4 = 2 - \tfrac{17}{4}\kappa^2\kappa'^2 \pm 2(1 - 2\kappa'^2)\,D,$$

$$\frac{5^3}{2^3}\,t^6 = 4 - \tfrac{51}{4}\kappa^2\kappa'^2 - \tfrac{19}{2}\kappa'^6 \pm (4 - \tfrac{49}{4}\kappa^2\kappa'^2)\,D,$$

$$\frac{5^4}{2^4}\,t^8 = 8 - 34\kappa'^2 + \tfrac{897}{16}\kappa'^4 - \tfrac{353}{8}\kappa'^6 + \tfrac{353}{16}\kappa'^8 \pm (8 - 25\kappa'^2 + 51\kappa'^4 - 34\kappa'^6)\,D.$$

On substitution I find, on putting $-f^2g^2\kappa^2\kappa'^2 = 1$,

$$\left.\begin{aligned}I_3^1(\sin) &= \frac{4\pi k^3\cos\beta\cos\gamma}{7\sin^3\beta}\cdot\frac{2^3}{3\,.\,5^4}\left[8 - 14\kappa^2\kappa'^2 + 3\kappa^4\kappa'^4 + (\kappa^2 - \kappa'^2)(8 + 3\kappa^2\kappa'^2)(1 - \tfrac{1}{4}\kappa^2\kappa'^2)^{\frac{1}{2}}\right]\\ I_3^3(\sin) &= \text{the same with the sign of the square root reversed.}\end{aligned}\right\} \quad . \quad (56)$$

Developing these expressions in powers of κ'^2, reintroducing the factor $-f^2g^2\kappa^2\kappa'^2$, and reverting to the notation of "Harmonics," I find

$$I_3^{\,1}(\sin) = \frac{4\pi k^3 \cos\beta\cos\gamma}{7\sin^3\beta} \cdot \frac{2^7}{3 \,.\, 5^4}\left(1 - \tfrac{7}{4}\kappa'^2 + \tfrac{169}{256}\kappa'^4\right)\left(-f^2g^2\kappa^2\kappa'^2\right)$$

$$= \frac{4\pi \mathrm{M}}{7} \cdot \frac{2^7}{3 \,.\, 5^4}\left(1 - \tfrac{7}{2}\beta + \tfrac{393}{64}\beta^2\right)\left(-f^2g^2\kappa^2\kappa'^2\right)$$

$$I_3^{\,3}(\sin) = \frac{4\pi k^3 \cos\beta\cos\gamma}{7\sin^3\beta} \cdot \tfrac{1}{2.5}\kappa'^4\left(1 - \tfrac{1}{2}\kappa'^2 + \tfrac{55}{256}\kappa'^4\right)\left(-f^2g^2\kappa^2\kappa'^2\right)$$

$$= \frac{4\pi \mathrm{M}}{7} \cdot \tfrac{1}{2.5}\kappa'^4\left(1 - \beta + \tfrac{119}{64}\beta^2\right)\left(-f^2g^2\kappa^2\kappa'^2\right).$$

In "Harmonics" I defined

$$\mathfrak{P}_3^{\,1}(\mu) = P_3^{\,1}(\mu) - \tfrac{1}{16}\beta\left(1 - \tfrac{3}{4}\beta\right)P_3^{\,3}(\mu)$$
$$= \tfrac{15}{2}\cos\theta\left[\sin^2\theta\left(1 + \tfrac{1}{8}\beta - \tfrac{3}{32}\beta^2\right) - \tfrac{1}{5}\left(1 + \tfrac{5}{8}\beta - \tfrac{15}{32}\beta^2\right)\right] \quad . \quad . \quad (57).$$

To make our former definition agree with this we must take

$$f\kappa \,.\, \kappa^2 = \tfrac{15}{2}\left(1 + \tfrac{1}{8}\beta - \tfrac{3}{32}\beta^2\right); \qquad f\kappa \,.\, q^2 = \tfrac{3}{2}\left(1 + \tfrac{5}{8}\beta - \tfrac{15}{32}\beta^2\right).$$

Hence $f\kappa = \tfrac{15}{2}\left(1 + \tfrac{17}{8}\beta + \tfrac{69}{32}\beta^2\right)$.
It will be found that $q^2 = \tfrac{1}{5}\left(1 - \tfrac{3}{2}\beta + \tfrac{9}{16}\beta^2\right)$, and that $f\kappa \,.\, q^2$ has the above form.
Again I defined

$$\mathfrak{S}_3^{\,1}(\phi) = \sin\phi - \tfrac{5}{8}\beta\left(1 - \tfrac{3}{4}\beta\right)\sin 3\phi,$$
$$= \sin\phi\left[1 - \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2 + \tfrac{5}{2}\beta\left(1 - \tfrac{3}{4}\beta\right)\sin^2\phi\right] \quad . \quad . \quad . \quad (58).$$

To make our former definition agree with this we must take

$$g\kappa'\sqrt{-1} \,.\, t^2 = 1 - \tfrac{15}{8}\beta + \tfrac{45}{32}\beta^2,$$

$$g\kappa'\sqrt{-1} \,.\, \kappa'^2 = \tfrac{5}{2}\beta\left(1 - \tfrac{3}{4}\beta\right).$$

It will be found that $t^2 = \tfrac{4}{5}\left(1 - \tfrac{17}{8}\beta + \tfrac{151}{64}\beta^2\right)$.
Whence $g\kappa'\sqrt{-1} = \tfrac{5}{4}\left(1 + \tfrac{1}{4}\beta - \tfrac{27}{64}\beta^2\right)$, and $g\kappa'\sqrt{-1} \,.\, \kappa'^2$ has the correct form.
Therefore

$$fg\kappa\kappa'\sqrt{-1} = \frac{3 \,.\, 5^2}{2^3}\left(1 + \tfrac{19}{8}\beta + \tfrac{145}{64}\beta^2\right)$$

$$-f^2g^2\kappa^2\kappa'^2 = \frac{3^2 \,.\, 5^4}{2^6}\left(1 + \tfrac{19}{4}\beta + \tfrac{651}{64}\beta^2\right).$$

Whence

$$I_3^{\,1}(\sin) = \frac{4\pi}{7}\mathrm{M} \,.\, 6\left(1 + \tfrac{5}{4}\beta - \tfrac{5}{16}\beta^2\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

agreeing with the result on p. 548 of "Harmonics" with $i = 3$, $s = 1$, type OOS.

In "Harmonics" I defined

$$\mathfrak{P}_3^3(\mu) = \tfrac{15}{4}\beta(1-\tfrac{3}{4}\beta)P_3^1(\mu) + P_3^3(\mu),$$
$$= 15\cos\theta[-(1-\tfrac{15}{8}\beta+\tfrac{45}{32}\beta^2)\sin^2\theta + 1 - \tfrac{3}{8}\beta + \tfrac{9}{32}\beta^2] \quad . \quad (60).$$

In order that this may agree with our former definition we must take

$$f\kappa\,.\,\kappa^2 = -15(1-\tfrac{15}{8}\beta+\tfrac{45}{32}\beta^2), \qquad f\kappa\,.\,q^2 = -15(1-\tfrac{3}{8}\beta+\tfrac{9}{32}\beta^2).$$

Whence $f\kappa = -15(1+\tfrac{1}{8}\beta-\tfrac{11}{32}\beta^2)$.
It will be found that

$$q^2 = 1 - \tfrac{1}{4}\kappa'^2 + \tfrac{3}{64}\kappa'^4 + \tfrac{3}{512}\kappa'^6 + \ldots$$
$$= 1 - \tfrac{1}{2}\beta + \tfrac{11}{16}\beta^2,$$

so that $f\kappa\,.\,q^2$ has the correct form.

Again I defined

$$\mathfrak{S}_3^3(\phi) = \tfrac{3}{8}\beta(1-\tfrac{3}{4}\beta)\sin\phi + \sin 3\phi,$$
$$= 3\sin\phi[1+\tfrac{1}{8}\beta-\tfrac{3}{32}\beta^2-\tfrac{4}{3}\sin^2\phi] \quad . \quad . \quad . \quad . \quad . \quad (61).$$

In order to make our former definition agree with this we must take

$$g\kappa'\sqrt{-1}\,.\,t^2 = 3(1+\tfrac{1}{8}\beta-\tfrac{3}{32}\beta^2), \qquad g\kappa'\sqrt{-1}\,.\,\kappa'^2 = -4.$$

Therefore $g\kappa'\sqrt{-1} = -\dfrac{4}{\kappa'^2}$.

It will be found that $t^2 = -\tfrac{3}{4}\kappa'^2(1+\tfrac{1}{8}\beta-\tfrac{3}{32}\beta^2)$, so that $g\kappa'\sqrt{-1}\,.\,t^2$ has the correct form.

Then

$$fg\kappa\kappa'\sqrt{-1} = -\frac{3\,.\,4\,.\,5}{\kappa'^2}(1+\tfrac{1}{8}\beta-\tfrac{11}{32}\beta^2),$$

and

$$-f^2g^2\kappa^2\kappa'^2 = \frac{3^2\,.\,4^2\,.\,5^2}{\kappa'^4}(1+\tfrac{1}{4}\beta-\tfrac{43}{64}\beta^2).$$

Whence

$$I_3^3(\sin) = \frac{4\pi M}{7}\,.\,360[1-\tfrac{3}{4}\beta+\tfrac{15}{16}\beta^2] \quad . \quad . \quad . \quad . \quad . \quad . \quad (62),$$

agreeing with the result on p. 548 of "Harmonics" with $i = 3$, $s = 3$, type OOS.

(5) *The Second Tesseral Sine Harmonic.*

This is defined thus:—

$$\left.\begin{aligned}\mathsf{P}_3^2(\mu) &= \sin\theta\cos\theta(1-\kappa^2\sin^2\theta)^{\frac{1}{2}} = f.\,\kappa\cos\theta(\kappa^2-\kappa^2\cos^2\theta)^{\frac{1}{2}}(\kappa'^2+\kappa^2\cos^2\theta)^{\frac{1}{2}},\\ \mathsf{S}_3^2(\phi) &= \sin\phi\cos\phi(1-\kappa'^2\cos^2\phi)^{\frac{1}{2}} = g.\,\kappa'\sqrt{-1}\sin\phi(\kappa^2+\kappa'^2\sin^2\phi)^{\frac{1}{2}}(\kappa'^2-\kappa'^2\sin^2\phi)^{\frac{1}{2}},\end{aligned}\right\}(63)$$

where $f = \dfrac{1}{\kappa^2}$, $g = \dfrac{1}{\kappa'^2\sqrt{-1}}$.

Squaring P_3^2 we have

$$A_0 = 0, \quad A_1 = \kappa^2\kappa'^2, \quad A_2 = \kappa^2 - \kappa'^2, \quad A_3 = -1.$$

Therefore

$$I_3^2(\cos) = \frac{4\pi k^3 \cos\beta\cos\gamma}{\sin^3\beta}\left[-\tfrac{1}{3}\kappa^6\kappa'^6 + \tfrac{1}{5}(\kappa^2-\kappa'^2)^2\kappa^4\kappa'^4 - \tfrac{1}{7}\kappa^6\kappa'^6 \right.$$
$$- \tfrac{4}{15}(\kappa^2-\kappa'^2)^2\kappa^4\kappa'^4 - \tfrac{6}{35}(\kappa^2-\kappa'^2)^2\kappa^4\kappa'^4$$
$$\left. + \tfrac{2}{105}(12 - 25\kappa^2\kappa'^2)\kappa^4\kappa'^4\right].f^2g^2.$$

Reducing this expression and putting $-f^2g^2\kappa^4\kappa'^4 = 1$, we have

$$I_3^2(\cos) = \frac{4\pi k^3\cos\beta\cos\gamma}{7\sin^3\beta}\cdot\tfrac{1}{3.5} \quad \ldots\ldots\ldots \quad (64).$$

In "Harmonics" I defined

$$\mathsf{P}_3^2(\mu) = \left(\frac{\frac{1+\beta}{1-\beta}-\mu^2}{1-\mu^2}\right)^{\frac{1}{2}} P_3^2(\mu) = 15\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\left(1-\frac{1-\beta}{1+\beta}\sin^2\theta\right)^{\frac{1}{2}}\cos\theta\sin\theta. \quad . \quad (65).$$

To make our former definition agree with this we must take

$$f\kappa^2 = 15\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}.$$

Again I defined

$$\mathsf{S}_3^2(\phi) = (1-\beta\cos 2\phi)^{\frac{1}{2}}\sin 2\phi = 2(1+\beta)^{\frac{1}{2}}\left(1-\frac{2\beta}{1+\beta}\cos^2\phi\right)^{\frac{1}{2}}\sin\phi\cos\phi. \quad (66).$$

To make the former definition agree with this we must take

$$g\kappa'^2\sqrt{-1} = 2(1+\beta)^{\frac{1}{2}}.$$

Therefore

$$fg\kappa^2\kappa'^2\sqrt{-1} = 2.3.5\frac{(1+\beta)}{(1-\beta)^{\frac{1}{2}}}, \text{ and}$$

$$-f^2g^2\kappa^4\kappa'^4 = 2^2.3^2.5^2\frac{(1+\beta)^2}{1-\beta} = 2^2.3^2.5^2(1+3\beta+4\beta^2).$$

Hence in the notation of "Harmonics"

$$I_3^2(\sin) = \frac{4\pi\mathrm{M}}{7}.3.4.5(1+3\beta+4\beta^2), \quad \ldots\ldots \quad (67)$$

agreeing with the result on p. 548 of "Harmonics" with $i = 3$, $s = 2$, type OES.

It may be convenient, as furnishing a kind of index to the foregoing investigation, to state that the 1 + 3 + 5 + 7 integrals for the harmonics of orders 0, 1, 2, 3 are given in equations 3, 5, 9, 13, 16, 24, 28, 32, 38, 47, 56, 64, corresponding to the definitions contained in 4, 8, 12, 15, 23, 27, 31, 36, 45, 54, 63.

The definitions of the harmonic functions as given in my paper on Harmonic Analysis are repeated in 6, 10, 19, 21, 25, 29, 34, 39, 40, 42, 43, 48, 49, 51, 52, 57, 58, 60, 61, 65, 66. Corresponding to these latter definitions the approximate integrals are given in 7, 11, 14, 20, 22, 26, 30, 35, 41, 44, 50, 53, 59, 62, 67; and the results confirm the correctness of the general approximate formulæ for the integrals given in § 22 of the paper on Harmonic Analysis.

A mistake in that paper was detected in the value of the cosine-function for the third zonal harmonic, and the corrected value is given in (40).

It must be obvious that the method exhibited here may be applied to higher harmonics with whatever degree of accuracy is desired; but it is also clear that the labour of evaluating the integrals increases very much as they rise in order. It is probable that the approximate results of the previous paper will suffice for most practical applications.

APPENDIX.

On the Symmetry of the Cosine and Sine-functions with the P-*functions.*

In my previous papers I failed to notice that the symmetry between the P-functions and the cosine and sine-functions is not destroyed, but is only masked, in the approximate expressions for the harmonic functions.

For example, (39) shows us that

$$\mathfrak{P}_3(\mu) = P_3(\mu) - \tfrac{1}{4}\beta P_3^2(\mu),$$

therefore, in consequence of the symmetry which subsists, we ought to find

$$\mathbf{C}_3(\phi) = P_3\left[\left(\frac{1-\beta\cos 2\phi}{1-\beta}\right)^{\frac{1}{2}}\right] - \tfrac{1}{4}\beta P_3^2\left[\left(\frac{1-\beta\cos 2\phi}{1-\beta}\right)^{\frac{1}{2}}\right].$$

Now
$$P_3\left[\left(\frac{1-\beta\cos 2\phi}{1-\beta}\right)^{\frac{1}{2}}\right] = \frac{(1-\beta\cos 2\phi)^{\frac{1}{2}}}{(1-\beta)^{3/2}}(1+\tfrac{3}{2}\beta-\tfrac{5}{2}\beta\cos 2\phi),$$

$$P_3^2\left[\left(\frac{1-\beta\cos 2\phi}{1-\beta}\right)^{\frac{1}{2}}\right] = -15\frac{(1-\beta\cos 2\phi)^{\frac{1}{2}}}{(1-\beta)^{3/2}}\beta(1-\cos 2\phi).$$

Whence
$$\mathbf{C}_3(\phi) = \frac{(1+\tfrac{3}{2}\beta+\tfrac{15}{4}\beta^2)}{(1-\beta)^{3/2}}(1-\beta\cos 2\phi)^{\frac{1}{2}}(1-\tfrac{5}{2}\beta\cos 2\phi).$$

This only differs by a constant factor from the expression (40).

It would be possible then to have only one type of function, viz. $\mathfrak{P}$ or $\mathbf{P}$, and to express all the cosine and sine-functions by means of the appropriate one of them. This would be found to be equivalent to expressing the latter functions in terms of powers of sin ϕ. For the purposes of practical application I do not think this would be so convenient as the use of cosines and sines of multiples of ϕ, and the advantage of using only one type of function would not compensate for the loss of convenience in the result. Accordingly I do not think it worth while to undertake the very laborious task of revising all the analysis of "Harmonics" from this point of view.

I may mention, however, that I have gone far enough in this direction to feel pretty confident that, if this new form of developing the cosine and sine-functions were adopted, the remarkable coincidence, mentioned in the footnote on p. 547 of "Harmonics," as to the form of the integrals of the squares of surface harmonics would become explicable.

POSTSCRIPT.

[*December* 24*th*, 1903.]

Mr. HOBSON has shown me how these integrals may be evaluated by a simpler method of analysis, without the intervention of elliptic integrals. As an example of the method he suggests I take the integral I_2 (cos) evaluated above.

The solid ellipsoidal harmonics are given, except as regards a factor, in § 3 of "The Pear-Shaped Figure."

In (19) of that paper we find

$$S_2 = \mathfrak{P}_2(\nu)\,\mathfrak{P}_2(\mu)\,\mathfrak{C}_2(\phi) = A\left[q^2x^2 + (1-2q^2)\,y^2 - q'^2z^2 + \frac{k^2}{\kappa^2}q^2q'^2\right],$$

where A is the factor to be evaluated so as to agree with the definitions

$$\mathfrak{P}_2(\nu) = \kappa^2\nu^2 - q^2, \qquad \mathfrak{P}_2(\mu) = \kappa^2\mu^2 - q^2, \qquad \mathfrak{C}_2(\phi) = q'^2 - \kappa'^2\cos^2\phi.$$

The ellipsoid over which we desire to integrate is defined by $\nu = \dfrac{1}{\kappa\sin\gamma}$, and the extremity of the c axis is defined by $\mu = \sin\theta = 1$, $\phi = \frac{1}{2}\pi$.

Hence at this point

$$S_2 = (\operatorname{cosec}^2\gamma - q^2)(\kappa^2 - q^2)\,q'^2.$$

But at the extremity of the c axis

$$x = 0, \qquad y = 0, \qquad z = c = \frac{k}{\kappa\sin\gamma}.$$

Therefore

$$(\operatorname{cosec}^2\gamma - q^2)(\kappa^2 - q^2)\,q'^2 = k^2A\left(-\frac{q'^2}{\kappa^2\sin^2\gamma} + \frac{q^2q'^2}{\kappa^2}\right) = -A\frac{k^2q'^2}{\kappa^2}(\operatorname{cosec}^2\gamma - q^2).$$

Therefore $A = -\dfrac{\kappa^2(\kappa^2 - q^2)}{k^2}$, and

$$S_2 = (\kappa^2 - q^2)\left[-q^2\kappa^2\frac{x^2}{k^2} - (1-2q^2)\,\kappa^2\frac{y^2}{k^2} + q'^2\kappa^2\frac{z^2}{k^2} - q^2q'^2\right].$$

Let us assume

$$x = a\xi = \frac{k \cos\gamma}{\kappa \sin\gamma}\,\xi, \qquad y = b\xi = \frac{k \cos\beta}{\kappa \sin\gamma}\,\eta, \qquad z = c\zeta = \frac{k}{\kappa \sin\gamma}\,\zeta.$$

Then when x, y, z is on the ellipsoid we have

$$\xi^2 + \eta^2 + \zeta^2 = 1.$$

Thus we may regard ξ, η, ζ as the coordinates of a point on a sphere of unit radius, or as direction cosines, if it is more convenient to do so. On substituting for x, y, z their values in terms of ξ, η, ζ we find

$$S_2 = (\operatorname{cosec}^2\gamma - q^2)(\kappa^2 - q^2)[- q^2\xi^2 - (1 - 2q^2)\eta^2 + q'^2\zeta^2].$$

On performing the same operation to the points on the boundary of an element $d\sigma$ of surface of the ellipsoid, we find

$$p\,d\sigma = abc\,d\omega = \frac{k^3 \cos\beta \cos\gamma}{\sin^3\beta}\,d\omega,$$

where $d\omega$ is an element of the surface of the sphere of unit radius, or an element of solid angle.

Since on the surface of the ellipsoid $\mathfrak{P}_2(\nu) = \operatorname{cosec}^2\gamma - q^2$, it follows that

$$\mathfrak{P}_2(\mu)\,\mathfrak{C}_2(\phi) = (\kappa^2 - q^2)[- q^2\xi^2 - (1 - 2q^2)\eta^2 + q'^2\zeta^2].$$

Hence

$$I_2(\cos) = \frac{k^3 \cos\beta \cos\gamma}{\sin^3\beta}(\kappa^2 - q^2)^2 \int [- q^2\xi^2 - (1 - 2q^2)\eta^2 + q'^2\zeta^2]^2\,d\omega.$$

It is easy to prove that

$$\int \xi^4\,d\omega = \int \eta^4\,d\omega = \int \zeta^4\,d\omega = \frac{4\pi}{5},$$

$$\int \eta^2\zeta^2\,d\omega = \int \zeta^2\xi^2\,d\omega = \int \xi^2\eta^2\,d\omega = \frac{4\pi}{15}.$$

Therefore

$$I_2(\cos) = \frac{4\pi k^3 \cos\beta \cos\gamma}{5 \sin^3\beta} \cdot \tfrac{1}{3}(\kappa^2 - q^2)^2 [3q^4 + 3(1 - 2q^2)^2 + 3q'^4 - 2q'^2(1 - 2q^2) + 2q^2(1 - 2q^2) - 2q^2q'^2]$$

$$= \frac{4\pi k^3 \cos\beta \cos\gamma}{5 \sin^3\beta} \cdot \tfrac{4}{3}(\kappa^2 - q^2)^2(1 - 3q^2q'^2).$$

On substituting for q^2 its value, viz., $\frac{1}{3}(1 + \kappa^2 - (1 - \kappa^2\kappa'^2)^{\frac{1}{2}})$, and effecting reductions we arrive at the result given in (16) above.

It is obvious that this process is considerably simpler and more elegant from the point of view of theory, but to carry these operations through for all the integrals given above would entail a good deal of algebra. I think indeed that the work might not be very much less than what I have already done.

Mr. HOBSON has further remarked that all the integrations may be avoided by the following theorem :—

If $F_n(\xi, \eta, \zeta)$ be a solid spherical harmonic function of ξ, η, ζ of degree n,

$$\int [F_n(\xi, \eta, \zeta)]^2 \, d\omega = 4\pi \frac{2^n \cdot n!}{2n+1!} F_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right) F_n(\xi, \eta, \zeta).$$

Considering, however, how simple are the integrals involved in his first method, it may be doubted whether this would save trouble.

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